NON-TRIVIAL MOTIVATIONAL CASE STUDIES FOR THE COLLEGE INTRODUCTORY COURSES IN MATHEMATICS

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A set of challenging motivational problems (with solutions) in Elementary Mathematics and Algebra has been discussed. The problems have been offered for students who are familiar with basic algebraic and geometrical concepts and are looking for modern applications of these concepts in advanced areas of mathematics (e.g., Theories of Numbers, Logics, and Algorithms) and computer science (e.g., Encryption Algorithms).

Introduction

The first course of Introductory Mathematics becomes a real challenge for many Rivier's freshman students. Some students have a weak background in Mathematics; some evening students took their last courses in high or middle schools several years ago and forgot even how to use simple arithmetic rules; and, finally, we have a group of advanced students, who are passionate for new knowledge, abstract mathematical concepts, and strong professional skills.

Traditional textbooks written for the courses of Introductory Mathematics [1, 2] contain basic algebraic principles, simple algorithms, "drilling" exercises, and trivial applications and examples that are mostly oriented on "below-average-level" students, giving no opportunities for advanced students in exploring the beauty of real abstract mathematical concepts and modern applications of these concepts in advanced areas of mathematics, computer science, networking technologies, biology, sociology, business, and other disciplines.

A different approach has been developed and introduced to Rivier's students, who love explorations in mathematics and accept the challenges of modern applications of basic mathematical concepts. We encourage students from local middle- and high-schools, college freshmen, and juniors in participating in regional, national, and (even) international contests in mathematics [3-9]; in taking our specially-designed course, "Modern Applications of High-School Mathematics," in the Rivier University Challenging Program; in participating in extracurricular activities, such as the Mathematical Seminar and the Students' Mathematical Club; and in exploring the challenging case studies offered in traditional classes.

In the present paper, we introduce several non-trivial motivational case studies that have been discussed recently in our Introductory Mathematics courses and the Challenging Program course. It was a challenge for the instructor to find the adequate problem prototypes in the literature. The popular mathematics journals, such as *The College*

Mathematics Journal [10] and *Mathematics Magazine* [11] offer mostly the advanced problems assuming the high-level mathematical background of readers. Finally, the instructor has referred to the problems from the American Mathematics Competitions (available via the Internet [4]) and unique books [8, 12] that he used in the past, teaching Advanced Mathematical Seminars for high-school students in Russia. A set of new motivational problems is offered here for readers and Rivier University students. Solutions to selected problems are also discussed.

1. Case Study 1: Restoring the Digits in Arithmetic Calculations with Multiplication of Large Numbers

1.1 Problem: Restore digits in the example below:

$$
\begin{array}{r}
* \, 1 \, * \, * \\
\times \quad 1 \, * \, * \\
\hline
* \, 1 \, * \, * \\
+ \quad * \, * \, * \, 1 \, * \\
* \, * \, * \, 1 \quad \\
\hline
8 \, * \, * \, 4 \, * \, *
\end{array}
\tag{1.1}
$$

Find all solutions.

1.2 Solutions

We have to find all possible values for the parameters a, b, c, d, e, f, g, h, i, j, k, m, n, p, q, r, s, t, and x in the following expression:

$$
\begin{array}{r}
a \, 1 \, b \, c \\
\times \quad 1 \, d \, e \\
\hline
f \, 1 \, g \, h \\
+ \quad i \, j \, k \, 1 \, x \\
m \, n \, p \, 1 \quad \\
\hline
8 \, q \, r \, 4 \, s \, t
\end{array}
\tag{1.2}
$$

We can find that (mnp1) = (a1bc) × 1 = (a1bc). Therefore, m = a; n = 1; p = b; and c = 1. Also, the only case is possible with e = 1. As a result, we can find: f = a; g = b; h = c = 1, and t =1 and x = d. After these substitutions to Eq. (1.2), we can find the following:

$$
\begin{array}{r}
a \, 1 \, b \, 1 \\
\times \quad 1 \, d \, 1 \\
\hline
a \, 1 \, b \, 1 \\
+ \quad i \, j \, k \, 1 \, d \\
a \, 1 \, b \, 1 \quad \\
\hline
8 \, q \, r \, 4 \, s \, 1
\end{array}
\tag{1.3}
$$

The last digit of the product b × d must be 1. The digital parameter s can be estimated from the formulas:

$$b + d = 10 + s; \quad b \times d = 10 \times w + 1 \tag{1.4}$$

There are only three cases that can satisfy the conditions of Eq. (1.4):

Case a): b = 9; d = 9, hence b × d = 81; b + d = 18, and s = 8; $\qquad$ (1.5a)

Case b): b = 3; d = 7, hence b × d = 21; b + d = 10, and s = 0; $\qquad$ (1.5b)

Case c): b = 7; d = 3, hence b × d = 21; b + d = 10, and s = 0. $\qquad$ (1.5c)

Let us study all these three cases in detail.

1.2.1 Case (a)

Under the conditions (1.5a), we can find from Eq. (1.3) the following:

```
      a 1 9 1
  ×       1 9 1
      a 1 9 1
  +  i j k 1 9
      a 1 9 1
  8 q r  4 8 1
```
(1.6a)

Here the parameter k must be equal to 7 only, and the following two relations have to be established between parameters a, i, and j:

$$10 \times i + j = 9 \times a + 1; \quad a \geq 1; \quad i > 0 \tag{1.7a}$$

$$7 \leq a + i \leq 8 \tag{1.8a}$$

These two conditions (1.7a, 1.8a) can be satisfied only at a = 4; therefore, i = 3 and j = 7. Parameters r and q can be found by substituting all other known digits (a, i, j, and k) into Eq. (1.6a). We find r = 0 and q = 0. Finally, in this case, the expressions (1.1) and (1.6a) can be reduced to Eq. (1.9a) below:

```
      4191
  ×    191
      4191
  +  37719
      4191
  800481
```
(1.9a)

1.2.2 Case (b)

Under the conditions (1.5b), we can find from Eq. (1.3) the following:

$$
\begin{array}{r}
a\,1\,3\,1 \\
\times\quad 1\,7\,1 \\
\hline
a\,1\,3\,1 \\
+\quad i\,j\,k\,1\,7 \\
\underline{a\,1\,3\,1\quad\;} \\
8\,q\,r\,4\,0\,1
\end{array}
\qquad\qquad (1.6b)
$$

Here the parameter k must be equal to 9 only, and the following two relations have to be established between parameters a, i, and j:

$$10 \times i + j = 7 \times a; \ \ a \geq 2; \ \ i > 0 \qquad\qquad (1.7b)$$

$$7 \leq a + i \leq 8 \qquad\qquad (1.8b)$$

These two conditions (1.7b, 1.8b) can be satisfied only at a = 5; therefore, i = 3 and j = 5. Parameters r and q can be found by substituting all other known digits (a, i, j, and k) into Eq. (1.6b). We find r = 7 and q = 7. Finally, in this case, the expressions (1.1) and (1.6b) can be reduced to Eq. (1.9b) below:

$$
\begin{array}{r}
5131 \\
\times\quad 171 \\
\hline
5131 \\
+\quad 35917 \\
\underline{5131\quad\;} \\
877401
\end{array}
\qquad\qquad (1.9b)
$$

1.2.3 Case (c)

Under the conditions (1.5c), we can find from Eq. (1.3) the following:

$$
\begin{array}{r}
a\,1\,7\,1 \\
\times\quad 1\,3\,1 \\
\hline
a\,1\,7\,1 \\
+\quad i\,j\,k\,1\,3 \\
\underline{a\,1\,7\,1\quad\;} \\
8\,q\,r\,4\,0\,1
\end{array}
\qquad\qquad (1.6c)
$$

Here the parameter k must be equal to 5 only, and the following two relations have to be established between parameters a, i, and j:

$$10 \times i + j = 3 \times a; \ \ a \geq 4; \ \ i > 0 \qquad\qquad (1.7c)$$

$$7 \le a + i \le 8 \qquad (1.8c)$$

These two conditions (1.7c & 1.8c) can be satisfied only at a = 6; therefore, i = 1 and j = 8. Parameters r and q can be found by substituting all other known digits (a, i, j, and k) into Eq. (1.6c). We find r = 8 and q = 0. Finally, in this case, the expressions (1.1) and (1.6c) can be reduced to Eq. (1.9c) below:

```
      6171
  ×    131
      6171                                    (1.9c)
  +  18513
     6171__
    808401
```

1.3 Answers:

a)
```
       4191
   ×    191
       4191
   +  37719
      4191__
     800481
```

b)
```
       5131
   ×    171
       5131
   +  35917
      5131__
     877401
```

b)
```
       6171
   ×    131
       6171
   +  18513
      6171__
     808401   ■
```

2. Case Study 2: Restoring the Digits in Arithmetic Calculations with Subtraction

2.1 Problem: Restore digits in the example below, where different letters indicate different digits and similar letters indicate similar digits:

$$\begin{array}{r} \text{ANNA} \\ - \quad \underline{\text{KATE}} \\ 4\ 7\ 0\ 9 \end{array} \qquad (2.1)$$

2.2 Solutions: Only three cases are candidates for the solutions:

$$K = 1, A = 6, N = 3, T = 2, E = 7; \qquad (2.2a)$$

$$K = 2, A = 7, N = 4, T = 3, E = 8; \qquad (2.2b)$$

$$K = 3, A = 8, N = 5, T = 4, E = 9. \qquad (2.2c)$$

2.3 Answers:

a)
$$\begin{array}{r} 6336 \\ - \quad \underline{1627} \\ 4709 \end{array}$$

b)
$$\begin{array}{r} 7447 \\ - \quad \underline{2738} \\ 4709 \end{array}$$

c)
$$\begin{array}{r} 8558 \\ - \quad \underline{3849} \\ 4709 \quad \blacksquare \end{array}$$

3. Case Study 3: Evaluating Expressions with Infinite Square-Root Expressions

3.1 Problem: Find the value of the expression:

$$\text{SQRT}(a \times \text{SQRT}(b \times (\text{SQRT}(a \times (\text{SQRT}(b \times (\text{SQRT}(a \times \text{SQRT}(b \times (\ldots)))))))))), \qquad (3.1)$$

where $a = 3^3$ and $b = 5^3$.

NOTE #1: There is infinite number of terms in the above expression.
NOTE #2: SQRT stands for the *Square-Root* function ($\sqrt{\ }$).

3.2 Solving Strategies

There are two strategies that can be explored by students. The first strategy is based on the fact that there is *infinite* number of terms in the given expression. Therefore, the equation (3.1) can be reformulated as the following:

$$X = \text{SQRT}(a \times \text{SQRT}(b \times (\text{SQRT}(a \times (\text{SQRT}(b \times (\text{SQRT}(a \times \text{SQRT}(b \times (\ldots)))))))))), \qquad (3.2)$$

$$X = \text{SQRT}(a \times \text{SQRT}(b \times X)) \tag{3.3}$$

The last transcendental equation (3.3) can be easily solved:

$$X^2 = a \times \text{SQRT}(b \times X); \qquad X^4 = a^2 \times b \times X; \qquad X^3 = a^2 \times b, \quad \text{and finally:}$$

$$X = a^{2/3} \times b^{1/3} \tag{3.4}$$

Therefore, after substituting $a = 3^3 = 27$ and $b = 5^3 = 125$ into Eq. (3.4), we find $X = 9 \times 5 = 45$, which is the answer to this problem.

The second strategy can be developed by using the geometric progression concept. The expression (3.2) can be written as the following:

$$X = a^S \times b^T \tag{3.5}$$

where

$$S = \tfrac{1}{2} + \tfrac{1}{2} \times \tfrac{1}{4} + \tfrac{1}{2} \times \tfrac{1}{4} \times \tfrac{1}{4} + \ldots + \tfrac{1}{2} \times (\tfrac{1}{4})^n + \ldots = \tfrac{1}{2} \div (1 - \tfrac{1}{4}) = \tfrac{2}{3} \tag{3.6}$$

$$T = \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{4} \times \tfrac{1}{4} + \ldots + \tfrac{1}{4} \times (\tfrac{1}{4})^n + \ldots = \tfrac{1}{4} \div (1 - \tfrac{1}{4}) = \tfrac{1}{3} \tag{3.7}$$

Therefore, $X = a^{2/3} \times b^{1/3}$, which is the equation (3.4).

3.3 Answer: $X = a^{2/3} \times b^{1/3}$. In the particular case, at $a = 3^3 = 27$ and $b = 5^3 = 125$, we find $X = 45$. ∎

4. Four Case Studies 4.1 – 4.4: Factor Analysis of Large Numbers

4.1 Problem: Prove that any six-digit number, which has the same three first and last digits (written in the same order), has factors (divisors) of 7, 11, and 13.

4.1.1 Solution to Problem 4.1. First of all, let's find the product of the factors, $7 \times 11 \times 13 = 1001$. Any six-digit number, which has the same three first and last digits (written in the same order), can be written in the following form:

$$abcabc = a \times 10^5 + b \times 10^4 + c \times 10^3 + a \times 10^2 + b \times 10^1 + c \tag{4.1a}$$

$$= a \times 10^2 \times (10^3 + 1) + b \times 10^1 \times (10^3 + 1) + c \times (10^3 + 1) \tag{4.1b}$$

$$= 1001 \times (100 \times a + 10^1 \times b + c). \tag{4.1c}$$

Therefore, the six-digit number, **abcabc** can be divided by $1001 = 7 \times 11 \times 13$, and numbers 7, 11, and 13 are the factors of the number, **abcabc**. ∎

4.2 Problem: Prove that $(2^{1776} - 1)$ can be divided by 7.

4.2.1 Solution to Problem 4.2. It is easier to find that the power parameter 1776 can be divided by three, $1776 = 3 \times 592$. Therefore, the number 2^{1776} can be written as the following:

$$2^{1776} = 8^{592} \tag{4.2a}$$

Using the factoring rule, $A^2 - B^2 = (A - B) \times (A + B)$, we can find the following:

$$2^{1776} - 1 = (8^{592} - 1) = (8^{296} - 1) \times (8^{296} + 1) = \dots$$
$$= (8^{37} - 1) \times (8^{37} + 1) \times (8^{74} + 1) \times (8^{148} + 1) \times (8^{296} + 1) \tag{4.2b}$$

At the next step, we can apply the Binomial Theorem to study the properties of the number 8^{37}:

$$(a + b)^n = a^n + n \times a^{n-1} \times b + n \times (n-1)/(2!) \times a^{n-2} \times b^2 + \dots + n \times a \times b^{n-1} + b^n \tag{4.2c}$$

In our case, $a = 7$, $b = 1$, and $n = 37$, and the number 8^{37} can be written as the following:

$$8^{37} = (7 + 1)^{37} = 7^{37} + 37 \times 7^{36} \times 1 + 37 \times 36/(2) \times 7^{35} \times 1 + \dots + 37 \times 7 \times 1 + 1 \tag{4.2d}$$

Therefore, the number $(8^{37} - 1)$ can be divided by 7, and the original number $(2^{1776} - 1)$ can be divided by 7 as well. ∎

4.3 Problem: Prove that the number $(\mathbf{n}^5 - 20\mathbf{n}^3 + 64\mathbf{n})$ can be divided by 3840. Here $\mathbf{n}$ is an even natural number and $\mathbf{n} > 4$.

4.3.1 Solution to Problem 4.3. By factoring, it is easier to estimate that $3840 = 2^8 \times 3 \times 5$. Using several times the basic factoring rule, $A^2 - B^2 = (A - B) \times (A + B)$, we can find the following:

$$\begin{aligned}
n^5 - 20n^3 + 64n &= n \times (n^4 - 20n^2 + 64) \\
&= n \times (n^2 - 16) \times (n^2 - 4) \\
&= (n-4) \times (n-2) \times n \times (n+2) \times (n+4)
\end{aligned} \tag{4.3a}$$

It is given that the parameter $\mathbf{n}$ is an even number and $\mathbf{n} > 4$; therefore, we can introduce a new parameter $k > 2$, such that $n = 2k$. After substituting $n = 2k$ to Eq. (4.3a), we find:

$$\begin{aligned}
n^5 - 20n^3 + 64n &= (2k-4) \times (2k-2) \times (2k) \times (2k+2) \times (2k+4) \\
&= 2^5 \times (k-2) \times (k-1) \times k \times (k+1) \times (k+2)
\end{aligned} \tag{4.3b}$$

The five subsequent numbers, $(k-2)$, $(k-1)$, k, $(k+1)$, and $(k+2)$ have numbers that can be divided by 2, 3, 4, and 5; therefore the product of these numbers can be divided by $2^3 \times 3$

× 5. Finally, the original number, $(\mathbf{n}^5 - 20\mathbf{n}^3 + 64\mathbf{n})$ can be divided by $3840 = 2^8 \times 3 \times 5$. ∎

4.4 Problem: Find the number of 0's in the expression:

$$\Pi = 1 \times 2 \times 3 \times \ldots \times 98 \times 99 \times 100 \qquad\qquad (4.4)$$

4.4.1 Solution to Problem 4.4. In other words, we have to estimate the number of factors 10 in the given expression (4.4a). The following properties and factors can be put into the consideration:

- $10 = 2 \times 5$
- There are 50 *even* numbers in the expression (4.4).
- Also, 20 numbers there can be divided by 5.
- In addition, 4 numbers (25, 50, 75, and 100) can be divided by 25.

Therefore, the expression (4.4) can be divided by 10^{24}. It means that the expression has 24 0's. ∎

5. Case Study 5: Estimating the Last Digit of the Large Number

We start every class with a brief discussion of an unusual non-trivial topic that is called a "warm-up" exercise [13]. After these "warm-up" exercises, the instructor offers a discussion on the main topic and asks students for a feedback on lecture materials and their arguments on selecting a competitive strategy for the problem analysis and development. These discussions help students to focus on the main point of the class session and stay active in class. Here is an example of the "warm-up" exercise that opens an introductory discussion of the theory of large numbers, which leads to the applied theory of encryption algorithms, such as the RSA Public-Key encryption algorithm [14]. At the same time, it illustrates a strong bond between mathematics and computer science. A student (even if he/she is not familiar with the theory of numbers) can solve the problem by a simple experimentation.

5.1 Problem: What is the Last Digit of the Number 2597^{5927} [mod(10)]?

5.2 Solution

We are interested in the last digit only of this number. Following the Newton's Binomial Theorem (see section 4.2.1), it is absolutely enough to consider the last digit of a simpler number 7^{5927}. Doing experiments with powers of number 7, we find that the last digit can only be 7, 9, 3, or 1, and therefore, it is a cycle of *four* cases. The power, 5927 can be represented as $5927 = 4 \times 1481 + 3$. Therefore, the last digit of 7^{5927} (and 2597^{5927}) is the same as the last digit of $7^3 = 343$, which is "**3**". Knowing two key parameters [e.g., the base (10) and the power (5927)], we can now restore all digits of the given huge number. ∎

6. Case Study 6: Applying Simple Trigonometric Formulas

6.1 Problem: Estimate the value of the expression:

$$\tan(1°) \times \tan(2°) \times \tan(3°) \times \ldots \times \tan(87°) \times \tan(88°) \times \tan(89°) \qquad (6.1)$$

NOTE: There are 89 terms in this expression.

6.2 Solution

The following simple trigonometric identities can be used in the analysis:

$$\tan(x) = \cot(90° - x), \text{ or } \tan(x) \times \tan(90° - x) = 1 \qquad (6.2)$$

After regrouping all the terms in Eq. (6.1) in pairs and apply the Eq. (6.2) to each pair, we will find the following:

$$[\tan(1°) \times \tan(89°)] \times [\tan(2°) \times \tan(88°)] \times \ldots \times [\tan(44°) \times \tan(46°)] \times \tan(45°)$$
$$= 1 \times 1 \times \ldots \times 1 \times \tan(45°) = \tan(45°) = 1 \qquad (6.3)$$

6.3 Answer:

$$\tan(1°) \times \tan(2°) \times \tan(3°) \times \ldots \times \tan(87°) \times \tan(88°) \times \tan(89°) = \mathbf{1.} \ \blacksquare$$

7. Case Study 7: "Minute" and "Hour" Clock-Hands in Coupling Positions

7.1 Problem: How many times a day (during 24 hours) "minute" and "hour" clock-hands take the same ("coupling") positions (h, m)? When this situation comes for the first time after the clock starts at 0:00? When the situation comes for the seventh time?

7.2 Solution

First of all, we have to find the "coordinates" of the "coupling" clock-hand positions (h, m) using the values (H, M) of the expression "$H_{hours}M_{minutes}$":

$$m = M; h = H + m/60 \qquad (7.1)$$

The hand rates can be estimated as following:

$$\text{hour-hand rate } r_h = 360°/12; \text{ minute-hand rate } r_m = 360°/60. \qquad (7.2)$$

The coupling φ-angular-based position criterion can be defined as:

$$\varphi_h = \varphi_m \ \underline{\text{or}} \ r_h h = r_m m \ \underline{\text{or}} \ h = m/5 \qquad (7.3)$$

Finally, the **general solution** can be found from Eqs. (7.1 & 7.3):

$$h = 12H/11; \tag{7.4}$$

$$m = M = 5h = 60H/11 \tag{7.5}$$

The calculated coupling clock-hand positions (h, m) are shown in the table below.

Coupling Clock-Hand Positions				
H, hours	M, minutes	h	m	Clock Reading*
0	0	0	0	12:00 AM
1	5.45	1.09	5.45	1:05 AM
2	10.91	2.18	10.91	2:11 AM
3	16.36	3.27	16.36	3:16 AM
4	21.82	4.36	21.82	4:22 AM
5	27.27	5.45	27.27	5:27 AM
6	32.73	6.55	32.73	6:33 AM
7	38.18	7.64	38.18	7:38 AM
8	43.64	8.73	43.64	8:44 AM
9	49.09	9.82	49.09	9:49 AM
10	54.55	10.91	54.55	10:55 AM
11	60.00	12.00	60.00	12:00 PM
1	5.45	1.09	5.45	1:05 PM
2	10.91	2.18	10.91	2:11 PM
3	16.36	3.27	16.36	3:16 PM
4	21.82	4.36	21.82	4:22 PM
5	27.27	5.45	27.27	5:27 PM
6	32.73	6.55	32.73	6:33 PM
7	38.18	7.64	38.18	7:38 PM
8	43.64	8.73	43.64	8:44 PM
9	49.09	9.82	49.09	9:49 PM
10	54.55	10.91	54.55	10:55 PM
11	60.00	12.00	60.00	12:00 AM
* NOTE: The clock readings are rounded to minutes.				

7.3 Answer:

Totally 23 times a day (24 hours) "*minute*" and "*hour*" clock-hands take the "coupling" positions. This situation comes for the *first* time (after the clock starts at 0:00 AM) at about 1:05 AM. The "coupling" event comes for the *seventh* time at about 7:38 AM. ∎

We invite readers to solve problems of case studies 8 and 9, and share their solutions with others.

8. Case Study 8: Logical Problem

8.1 Problem: In the in-class game, all students have been divided into two teams: "serious students", who are answering *correctly* to any question, and "jokers", who only are answering *incorrectly* to any question. A teacher has asked Smith is he a "serious" student or a "joker". The teacher did not hear the Smith's answer well, and he has asked both Parker and Dennis the question: "What did Smith answer to me?" Parker has told: "The Smith's answer was being that he is a "serious student." Dennis has told: "The Smith's answer was being that he is a "joker." Who are Parker and Dennis: "serious students" or "jokers"?

9. Case Study 9: Combinatorics

9.1 Problem: The hockey-on-ice team includes three forwarders, two defenders, and one goalkeeper. How many different on-ice teams the hockey team trainer can create, if the Rivier University team has 7 forwarders, 5 defenders, and 2 goalkeepers?

10. Conclusion

Several specially-designed case studies have been used for motivating students (both in high schools and colleges) in exploring the advanced topics of Elementary Mathematics and Algebra. This approach allows the students to build strong analytical skills and search for modern applications of these topics in various areas of mathematics and computer science.

References

[1] Bello, I. *Introductory Algebra: A Real World Approach*, 2nd edition, New York: McGraw-Hill, 2006.
[2] Bittenger, M.L. *Introductory Algebra*, 8th edition. Reading, MA: Addison-Wesley, 1998.
[3] *The North Carolina State High School Mathematics Contest*, North Carolina Council of Teachers of Mathematics. [Online] http://courses.ncssm.edu/goebel/statecon/state.htm.
[4] *The American Mathematics Competitions*, University of Nebraska-Lincoln. [Online] http://www.unl.edu/amc/.
[5] Zare, D. *Internet Mathematics Contest*, Columbia University. [Online] http://www.math.columbia.edu/~zare/contest.html.
[6] *Annual International Mathematics Contest for Junior, Intermediate and Senior High Schools*, American Scholastic Mathematics Association. [Online] http://www.asan.com/asa/asma1.htm

[7] *International Mathematics Competition for University Students (IMC)*. [Online] http://www.imc-math.org/.

[8] Galperin, G.A., & Tolpyego, A.K. *Moscow Mathematics Contests: The Book for School Students*, edited by A. N. Kolmogorov, Moscow, Russia: Prosvescshenie, 1986 (in Russian).

[9] Sergeev, I.N. (editor). *International Mathematics Contests*. The Library of [School] Mathematical Club, Vol. 17, Moscow, Russia: Nauka, 1987 (in Russian).

[10] Problems and Solutions, *The College Mathematics Journal*, 2006; 37(4):308-318.

[11] Problems: Proposals, Quickies, Solutions, and Answers, *Mathematics Magazine*, 2006; 79(4):309-317.

[12] Petrakov, I.S. *Extracurricular Mathematical Seminars in High Schools: The Book for Teachers*, Moscow, Russia: Prosvescshenie, 1987 (in Russian).

[13] Riabov, V.V. Challenging Projects and Virtual Labs in Web-enhanced Networking Technology Classes. *Journal of Computing Sciences in Colleges*, 2006; 21(6):88-99.

[14] Rivest, R., Shamir, A., & Adleman, L. A Method for Obtaining Digital Signatures and Public Key Cryptosystems. In: *Communications of the ACM*, February 1978.