OPTIMAL STRATEGIES FOR FINDING MAXIMUM

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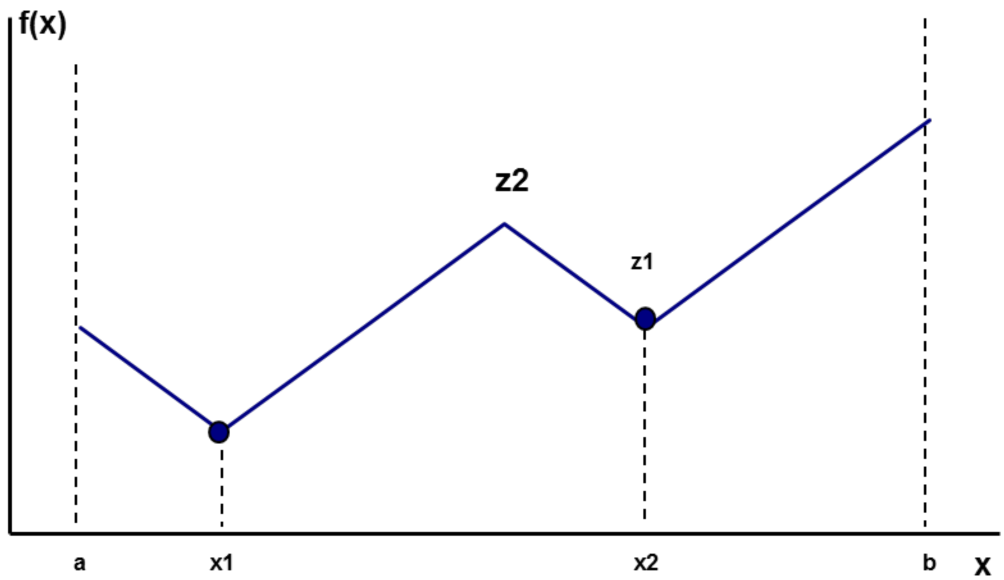
In this paper we will develop a numerical method to determine the maximum value of a function. The only a priori information we have is that the function belongs to a certain class. We will consider the class of functions with bounded first derivatives such that $|f'(x)| \le M$ for some constant real number *M*. Our numerical method will consist of evaluating the function at certain points, and estimating its maximum based upon this information. We will develop two separate methods-- one non-adaptive, where all the points are chosen at once at the beginning, and one adaptive, where each next point is selected based on the information we have already collected. Given a series of points on an interval [*a*,*b*], we can compute the radius of information (*R*(*I*)) using the formula $R(I) = \frac{z_2 - z_1}{2}$. In this formula z_2 is the maximum possible value of a function passing through those points, and z_1 is the minimum possible maximum, that is, any function passing through the given points can have a maximum of no less than z_1. It is clear that z_1, then, is simply the greatest given *y* value. An example is given below:



Using the ideas of game theory, we can determine which choices of *x* will minimize this value. We can treat this problem as a game in which nature plays against us, where we can choose the *x* values of a set of points (a strategy), while nature chooses the corresponding *y* values. Our goal is to minimize the radius of information, while nature will be trying to maximize *R*(*I*). In game theory, an optimal strategy is one for which any deviation leads to a loss of value for the player. An optimal strategy for nature maximizes *R*(*I*) over *y*, so our optimal strategy is equivalent to $\min_{x_1,\ldots,x_n} \max_{y_1,\ldots,y_n} R(I)$.

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The first case we shall consider is when there is a limitation on the first derivative such that $|f'(x)| \le M$ for some constant real number *M*. Out method in this case is equally applicable to the class of Lipschitz functions, which are functions such that $|f(x) - f(y)| \le M * |x - y|$ for all *x* and *y*, where *M* is a constant. We can devise two separate strategies—one in which we choose all the *x* values at once (non-adaptive), and one in which we choose only the next *x* value if we are already given a set of points (one-step optimal).

Non-adaptive Strategy

1. Theorem: The strategy $Y = \{y_1, \ \ldots, \ y_n\}$ where $y_i = C$ for *i* = 1, 2, …, *n* and *C* is a constant real number is optimal. That is, it maximizes *R*(*I*).

Proof: Let x_1 and x_2 be two successive *x* values, and *C* be the value of the greatest *y*. If the slope has the limitation *M*, then the maximum value the function can possibly take on the interval $[x_1, x_2]$ is given by the intersection of the lines $y = Mx + y_1 - Mx_1$ and $y = -Mx + y_2 + Mx_2$. Thus

$x = \dfrac{y_2 - y_1 + Mx_2 + Mx_1}{2M}$ and the intersection is

$y = \dfrac{y_2 - y_1 + Mx_2 + Mx_1}{2} + y_1 - Mx_1 = \dfrac{y_2 + y_1 + Mx_2 - Mx_1}{2}$. The potential *R*(*I*) on this interval is $\dfrac{y_2 + y_1 + Mx_2 - Mx_1}{4} - \dfrac{2*C}{4}$. This is maximized when $y_1 = y_2 = C$ and $R(I) = \dfrac{M(x_2 - x_1)}{4}$. Thus the *y* values of any two successive points should be equal, and the optimal strategy is $y_i = C$ for *i* = 1, 2, …, *n* where *C* is a constant real number.■

2. Theorem: The strategy $X = \{x_1, \ \ldots, \ x_n\}$ where $x_i = \dfrac{2an + 2bi - 2ai + a - b}{2n}$ for *i* = 1, 2, …, *n* is optimal. That is, it minimizes $\max\limits_{y_1,\ldots,y_n} R(I)$.

Proof: Assume the worst case scenario where nature plays optimally—that is, where *R*(*I*) is maximized over $y_1,\ldots,y_n$. Then $Y = \{y_1, \ \ldots, \ y_n\}$ where $y_i = D$ for *i* = 1, 2, …, *n* and *D* is a constant real number. Let $X = \{x_1, \ \ldots, \ x_n\}$ where $x_i = \dfrac{2an + 2bi - 2ai + a - b}{2n}$ for *i* = 1, 2, …, *n*, be given. If *M* is the limitation on the slope, then $R(I) = M * \max\{x_1 - a, b - x_n, \dfrac{x_{k+1} - x_k}{2} \mid k \in \{1,\ldots,n-1\}\}/2$. Thus with the given strategy we have $x_1 - a = \dfrac{2an + 2b - 2a + a - b}{2n} - a = \dfrac{2an + b - a}{2n} - \dfrac{2an}{2n} = \dfrac{b - a}{2n}$,

$b - x_n = b - \dfrac{2an + 2bn - 2an + a - b}{2n} = b - \dfrac{2bn + a - b}{2n} = \dfrac{b - a}{2n}$, and

$\dfrac{x_{k+1} - x_k}{2} = \dfrac{1}{2} * \left(\dfrac{2an + 2b(k+1) - 2a(k+1) + a - b}{2n} - \dfrac{2an + 2bk - 2ak + a - b}{2n} \right) =$

$\frac{2b(k+1)-2a(k+1)-2bk+2ak}{4n}=\frac{2b(k+1-k)-2a(k+1-k)}{4n}=\frac{2b-2a}{4n}=\frac{b-a}{2n}$ for all $k\in\{1,\ldots,n-1\}$.

Therefore $R(I)=M*\frac{b-a}{2n}*\frac{1}{2}=\frac{M(b-a)}{4n}$.■

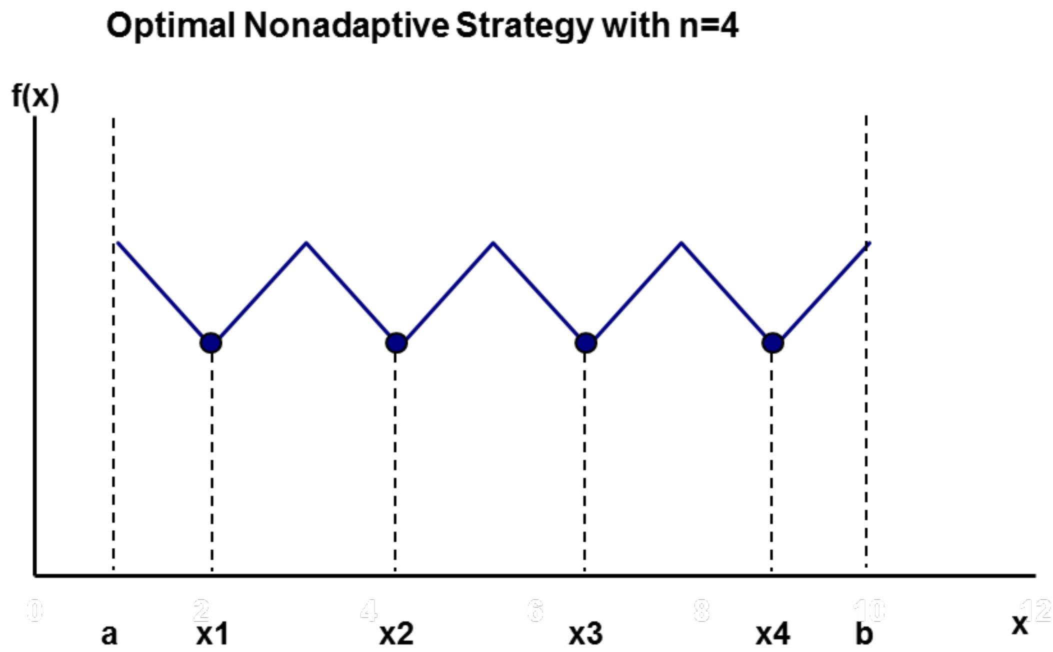
We will show by cases that shifting any x_i will increase the value of $R(I)$, and thus the given strategy is optimal.

Case 1: We shift x_i to the right by adding a real number C. We see that for x_1, $(x_1+C)-a=\frac{b-a}{2n}+C>\frac{b-a}{2n}$. Thus $R(I)=\frac{M(b-a+2nC)}{4n}>\frac{M(b-a)}{4n}$. For any other x_i (x_j for j = 2, …, n) we have $\frac{(x_j+C)-x_{j-1}}{2}=\frac{1}{2}*\left(\frac{b-a}{n}+C\right)=\frac{b-a+nC}{2n}$. Hence $R(I)=\frac{M(b-a+nC)}{4n}>\frac{M(b-a)}{4n}$. Thus shifting any of the given x values to the right will increase the radius of information.

Case 2: We shift x_i to the left by subtracting a real number C. For x_n, $b-(x_n-C)=b-x_n+C=\frac{b-a}{2n}+C$. Thus $R(I)=\frac{M(b-a+2nC)}{4n}>\frac{M(b-a)}{4n}$. For any other x_i (x_k for k = 1, …, n-1) we have $\frac{x_{k+1}-(x_k-C)}{2}=\frac{x_{k+1}-x_k+C}{2}=\frac{1}{2}*\left(\frac{b-a}{n}+C\right)=\frac{b-a+nC}{2n}$. Hence $R(I)=\frac{M(b-a+nC)}{4n}>\frac{M(b-a)}{4n}$. Thus shifting any of the given x values to the left will also increase the radius of information.

Since we cannot shift any value, either to the right or to the left, without increasing the radius of information, the strategy $X=\{x_1, \ldots, x_n\}$ where $x_i=\frac{2an+2bi-2ai+a-b}{2n}$ for i = 1, 2, …, n is optimal.

3. By the preceding proof, $\min_{x_1,\ldots,x_n}\ \max_{y_1,\ldots,y_n} R(I)=\frac{M(b-a)}{4n}$.



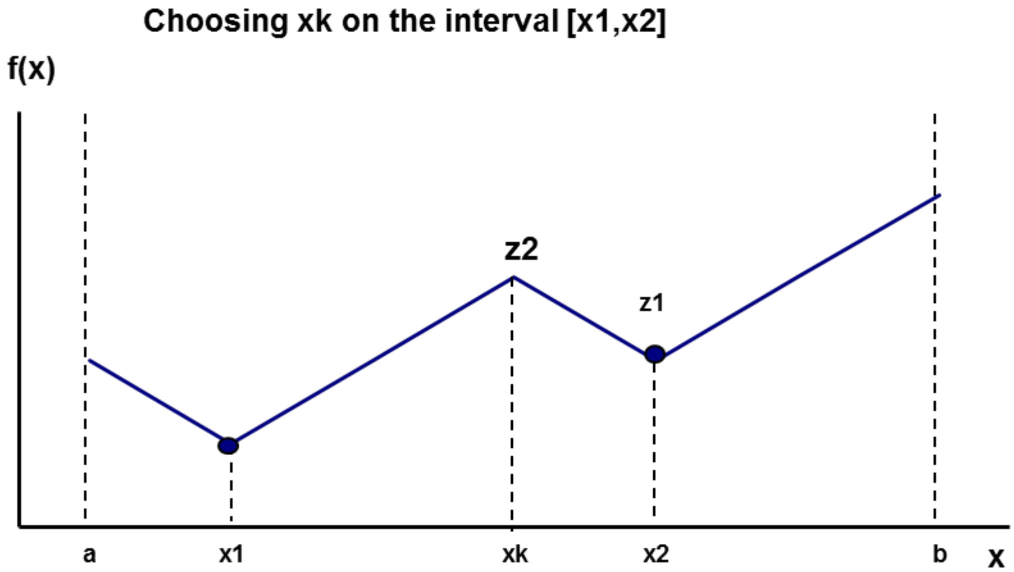
One-step Optimal Strategy

Given the interval $[a,b]$, the limitation on the slope M, and a set of points, we want to determine the optimal placement of another x value, x_k, to minimize $R(I)$. If x_k is not placed on the same interval as z_2, then $R(I)$ will not be lowered. Hence x_k must be chosen on the same interval as z_2.

1. Theorem: If z_2 occurs on the interval $[x_1, x_2]$ where x_1 and x_2 are two successive x values, then the optimal choice of x_k is $x_k = \dfrac{y_2 - y_1 + M(x_1 + x_2)}{2M}$.

Proof: Let z_2 occur on the interval $[x_1, x_2]$ where x_1 and x_2 are two successive x values. We want to minimize z_2 over this interval with our choice of x_k, which is equivalent to minimizing z_2 over the intervals $[x_1, x_k]$ and $[x_k, x_2]$. Let y_L be the maximum possible maximum on $[x_1, x_k]$, with y_R on $[x_k, x_2]$. We see that $y_1 = Mx_1 + (y_1 - Mx_1)$, $y_k = -Mx_k + (y_k + Mx_k)$, and their intersection is $Mx + (y_1 - Mx_1) = -Mx + (y_k + Mx_k)$, $x = \dfrac{y_k - y_1 + Mx_k + Mx_1}{2M}$, and $y_L = \dfrac{y_k - y_1 + Mx_k + Mx_1}{2} + y_1 - Mx_1 = \dfrac{y_k + y_1 + Mx_k - Mx_1}{2}$. Similarly $y_R = \dfrac{y_2 + y_k + Mx_2 - Mx_k}{2}$. To minimize $\max\{y_L, y_R\}$ we must have $y_L = y_R$,

$$y_k + y_1 + Mx_k - Mx_1 = y_2 + y_k + Mx_2 - Mx_k, \quad x_k = \frac{y_2 - y_1 + M(x_1 + x_2)}{2M}. \blacksquare$$



1.2. Theorem: The new radius of information on this interval will be at most $\dfrac{y_2 - y_1 + Mx_2 - Mx_1}{8}$ if $y_1 \geq y_2$, and at most $\dfrac{y_1 - y_2 + Mx_2 - Mx_1}{8}$ if $y_1 \leq y_2$.

Proof: $z_2 = \dfrac{y_k + y_1 + Mx_k - Mx_1}{2} = \dfrac{2y_k + y_1 + y_2 + Mx_2 - Mx_1}{4}$. The worst case scenario is when nature plays optimally and $y_k = z_1 = \max\{y_1, y_2\}$. If $y_1 \geq y_2$, then $R(I) = \dfrac{3y_1 + y_2 + Mx_2 - Mx_1 - 4y_1}{8} = \dfrac{y_2 - y_1 + Mx_2 - Mx_1}{8}$. If $y_1 \leq y_2$, then $R(I) = \dfrac{y_1 - y_2 + Mx_2 - Mx_1}{8}$. ■

2. Theorem: If z_2 occurs on the interval $[a, x_1]$ where x_1 is the smallest x value, then the optimal choice of x_k is $x_k = \dfrac{x_1 + 2a}{3}$.

Proof: Let z_2 occur on the interval $[a, x_1]$ where x_1 is the least x value. We want to minimize z_2 over the intervals $[a, x_k]$ and $[x_k, x_1]$. Let y_L be the maximum possible maximum on $[a, x_k]$, with y_R on $[x_k, x_1]$. Then on the left $y_k = -Mx_k + (y_k + Mx_k)$, $y_L = -Ma + y_k + Mx_k$. Also $y_R = \dfrac{y_1 + y_k + Mx_1 - Mx_k}{2}$. If $y_L = y_R$, then $-2Ma + 2y_k + 2Mx_k = y_1 + y_k + Mx_1 - Mx_k$,

$3Mx_k = y_1 - y_k + Mx_1 + 2Ma$, and $x_k = \dfrac{y_1 - y_k + Mx_1 + 2Ma}{3M}$. Assuming the worst case scenario where nature plays optimally, $y_k = y_1$. Thus $x_k = \dfrac{Mx_1 + 2Ma}{3M} = \dfrac{x_1 + 2a}{3}$. ■

2.2. Theorem: The new radius of information on this interval will be at most $\dfrac{Mx_1 - Ma}{6}$.

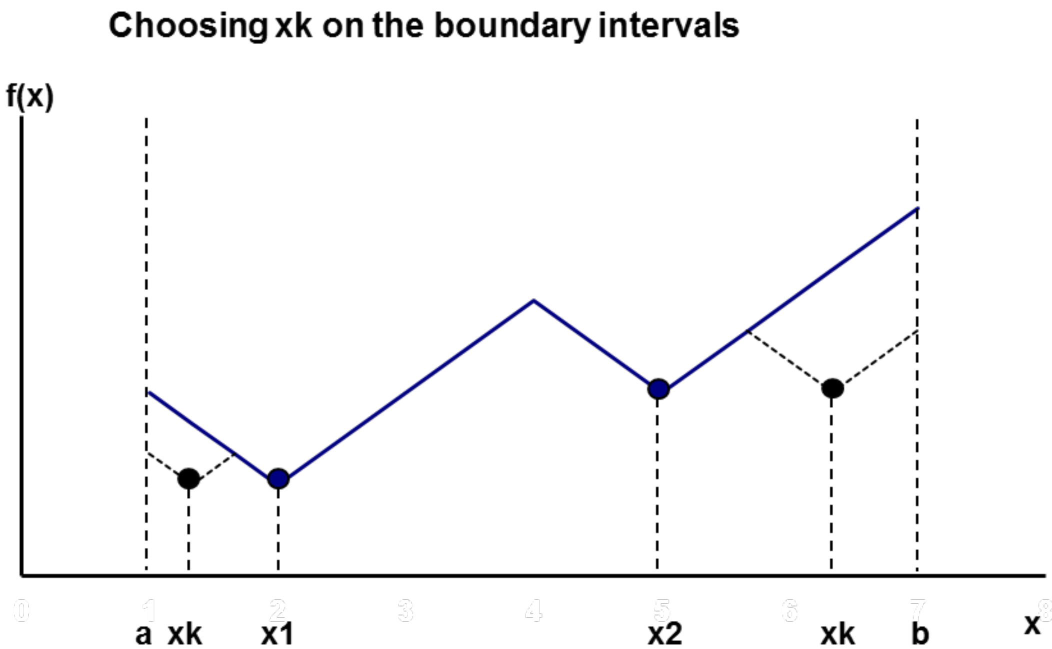
Proof: $z_2 = -Ma + y_k + M\left(\dfrac{x_1 + 2a}{3}\right) = \dfrac{3y_k + Mx_1 - Ma}{3}$. The worst case is when nature plays optimally and $y_k = y_1 = z_1$, and $R(I) = \dfrac{3y_k + Mx_1 - Ma - 3z_1}{3*2} = \dfrac{Mx_1 - Ma}{6}$. ■

3. Theorem: If z_2 occurs on the interval $[x_1, b]$ where x_1 is the greatest *x* value, then the optimal choice of x_k is $x_k = \dfrac{x_1 + 2b}{3}$.

Proof: Let z_2 occur on the interval $[x_1, b]$ where x_1 is the greatest *x* value. We want to minimize z_2 over the intervals $[x_1, x_k]$ and $[x_k, b]$. Let y_L be the maximum possible maximum on $[x_1, x_k]$ with y_R on $[x_k, b]$. Then $y_L = \dfrac{y_k + y_1 + Mx_k - Mx_1}{2}$. On the right $y_k = Mx_k + (y_k - Mx_k)$, $y_R = Mb + y_k - Mx_k$. If $y_L = y_R$, then $2Mb + 2y_k - 2Mx_k = y_k + y_1 + Mx_k - Mx_1$, and $x_k = \dfrac{y_k - y_1 + Mx_1 + 2Mb}{3M}$. Assuming the worst case scenario where nature plays optimally, $y_k = y_1$. Thus $x_k = \dfrac{Mx_1 + 2Mb}{3M} = \dfrac{x_1 + 2b}{3}$. ■

3.2. Theorem: The new radius of information on this interval will be at most $\dfrac{Mb - Mx_1}{6}$.

Proof: $z_2 = Mb + y_k - M\left(\dfrac{x_1 + 2b}{3}\right) = \dfrac{3y_k + Mb - Mx_1}{3}$. The worst case scenario is when nature plays optimally and $y_k = y_1 = z_1$, and $R(I) = \dfrac{Mb - Mx_1}{6}$. ■



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